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ROUGH QUASI-IDEALS IN REGULAR SEMI RINGS

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Abstract

In this paper we study the notion of quasi-ideals and quasi-k-ideals in regular semirings and characterize quasi-ideals and quasi-k-ideals in terms of the rough approximations. Finally it is shown that a semiring R is regular iff $\theta(\widehat{A)\theta(}B) = \theta(A) \cap \theta(B)$ for any right k-ideal A and left k-ideal B of R and it is established that R is regular and $\theta(Q) = \theta(\widehat{Q)R\theta}(Q)$ are equivalent in semirings.

1.Introduction

Rough set theory [12-15] is a mathematical approach to deal with inexact, uncertain or vague knowledge. It has recently received wide attention on the research areas in both the real-life applications and the theory itself. Rough set theory is an extension of set theory, in which a subset of a universe is described by a pair of ordinary sets called the lower and upper approximations. Rough sets are a suitable mathematical model for vague concepts. Rough set theory is emerging as a powerful tool to deal with imperfect data. It has found practical applications in many areas such as knowledge discovery, machine learning, data analysis, data mining, approximate classification and so on. Basic ideas of the rough set theory was introduced by Pawlak[12] in 1982. Kuroki[8] introduced the notion of a rough ideal in a semigroup. Kuroki and Wang[9] gave some properties of the lower and upper approximations with respect to the normal subgroups. Davvaz[4,5] gave the relationship between rough set and ring theory. He considered a ring as the universal set and introduced the notion of rough ideals and rough subrings with respect to an ideal of a ring.

The aim of this paper is to bring out the relationship between rough set theory and semirings as well as to widen the applicability of rough set theory to more algebraic structures. In this paper we study the notion of quasi-ideals and quasi-k- ideals in regular semirings. The rest of the paper is organized as follows. Section 2 contains some preliminaries. Section 3 deals with quasi-ideals and quasi-k-ideals in semirings. We characterize quasi-ideals and quasi-k-ideals in terms of lower and upper approximations and it is shown that a semiring R is regular iff $\theta(\widehat{A)\theta(}B) = \theta(A) \cap \theta(B)$ for any right k-ideal A and left k-ideal B of R and it is established that R is regular and $\theta(Q) = \theta(\widehat{Q)R\theta}(Q)$ are equivalent in semirings. In section 4 a brief conclusion is presented.

2 Preliminaries and Congruence Relation

In this section some definitions and results are reproduced, which are proposed by pioneers in this field earlier and are necessary for the development of the new results.

A *semiring* is a system consisting of a nonempty set R together with binary operations on R called addition and *multiplication* such that $(R,+)$ is a semigroup; $(R,.)$ is a semigroup and $a.(b+c) = a.b + a.c$ and $(b+c).a = b.a + c.a$ for all $a,b,c \in R$. A semiring R may have an *identity 1*,defined by $1.a = a = a.1$ and a zero 0,defined by $0 + a = a = a + 0$ and $a.0 = 0 = 0.a$ for all $a \in R$. From now on we write R for semirings. A nonempty subset I of R is said to be a *left*(resp. *right*) *ideal* if $x,y \in I$ and $r \in R$ imply that $x + y \in I$ and $r.x \in I$ and (resp. $x.r \in I$). I is said to be *two –sided ideal* or simply *ideal* of R, if I is both left and right ideal of R.

A left ideal I of a semiring R is said to be a *left k-ideal* if $a \in I, x \in R$ and $a + x \in I$ or $x + a \in I$, then $x \in I$. A *right k-ideal* is defined dually, and a *two sided k-ideal* or simply a *k-ideal* is both a left and a right k-ideal. The ideal generated by $a \in R$ is defined as the smallest ideal of R, which contains a and is denoted by $< a >$. The k-ideal generated by $a \in R$ is defined as the smallest k-ideal of R, which contains a and is denoted by $< a >_k$.

Let $a \in R$. We denote by $I(a)$ (resp., $L(a), R(a), L_k(a), R_k(a)$) the *ideal*,(resp., *left ideal*, *right ideal*, *left k-ideal*, *right k-ideal*) of R generated by $a \in R$. One can easily prove that

$I(a) = \{a + sa + at + s_i a t_i | \, s, t, s_i, t_i \in R\}$; $L(a) = \{a + sa | \, s \in R\}$; $R(a) = \{a + at | \, t \in R\}$

$L_k(a) = \{u \in R | u + a + sa = a + sa \; for \; some \; s \in R\}$

$R_k(a) = \{u \in R | u + a + sa = a + as \; for \; some \; s \in R\}$

For the sake of convenience we write ab instead of $a \cdot b$.

Let U be a universal set. An *equivalence relation* θ on R is a reflexive, symmetric and transitive binary relation on U. The set of elements of U that are related to $x \in U$ is called the equivalence class of x and is denoted by $[x]_\theta$.

Definition 2.1 A pair (U, θ) wher $U \neq \phi$ and θ is an equivalence relation on U, is called an *approximation space*

Definition 2.2 For an approximation space (U, θ) by a *rough approximation* in (U, θ) we mean a mapping $\rho: \wp(U) \to \wp(U) \times \wp(U)$ defined as $\rho(X) = \left(\underline{\rho}(X), \overline{\rho}(X)\right)$ for $X \subseteq U$ where $\underline{\rho}(X) = \{x \in U | [x]_\theta \subseteq U\}$, and $\overline{\rho}(X) = \{x \in U | [x]_\theta \cap U \neq \phi\}$, $\underline{\rho}(X)$ is called a *lower rough approximation* of X in (U, θ) where as $\overline{\rho}(X)$ is called an *upper rough approximation* of X in (U, θ).

Hereafter we use $\underline{\theta}(X), \overline{\theta}(X)$ and $\theta(X)$ instead of $\underline{\rho}(X), \overline{\rho}(X)$ and $\rho(X)$ respectively.

Definition 2.3 Given an approximation space (U, θ) a pair $(A, B) \in \wp(U) \times \wp(U)$ is called a *rough set* in (U, θ) if and only if $(A, B) = \theta(X)$ for some $X \subseteq U$.

If A nd B are any two subsets of R, then $AB = \{ab | \, a \in A, b \in B\}$.

Definition 2.4 Let θ be an equivalence relation on R. θ is called a *congruence relation* if $(a, b) \in \theta$ implies

(i) $(a + x, b + x) \in \theta$; (ii) $(x + a, x + b) \in \theta$; (iii) $(ax, bx) \in \theta$ and (iv) $(xa, xb) \in \theta$, for all $x \in R$.

The following theorem is an immediately consequence of Definition 2.4.

Theorem 2.5. *Let θ be a congruence relation on a semiring R.* Then $(a, b), (c, d) \in \theta$ implies $(a + c, b + d) \in \theta$, $(ac, bd) \in \theta$ for all $a, b, c, d \in R$.

Lemma 2.6. *Let θ be a congruence relation on R. If $a, b \in R$, then*

(i) $[a]_\theta + [b]_\theta \subseteq [a + b]_\theta$

(ii) $[a]_\theta . [b]_\theta \subseteq [ab]_\theta$

Proof. (i) Let $x \in R$. Suppose $x \in [a]_\theta + [b]_\theta$. Then there exist $y, z \in R$ such that $y \in [a]_\theta, z \in [b]_\theta$ and $x = y + z$. This means that $(a, y), (b, z) \in \theta$ and hence $(a + b, y + z) = (a + b, x) \in \theta$. Thus $x \in [a + b]_\theta$ and hence $[a]_\theta + [b]_\theta \subseteq [a + b]_\theta$.

(ii)Let $z = xy \in [a]_\theta . [b]_\theta$. Then $x \in [a]_\theta$ and $y \in [b]_\theta$. This implies that $(a, x) \in \theta$ and $(b, y) \in \theta$. Since θ is a congruence relation, $(ab, xy) \in \theta$.Thus $z = xy \in [ab]_\theta$ and hence $[a]_\theta . [b]_\theta \subseteq [ab]_\theta$.

A congruence relation θ on R is called complete if $[a]_\theta + [b]_\theta = [a + b]_\theta$ and $[a]_\theta . [b]_\theta = [ab]_\theta$.

Definition 2.7. Let θ be a congruence relation on R and A a subset of R. Then the sets

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$\underline{\theta}(\mathrm{A}) = \{\mathrm{x} \in \mathrm{R}/[\mathrm{x}]_\theta \subseteq \mathrm{A}\}$ and $\overline{\theta}(\mathrm{A}) = \{\mathrm{x} \in \mathrm{S}/[\mathrm{x}]_\theta \cap \mathrm{A} \neq \phi\}$ are called the lower and upper approximations of the set A, respectively. Let A be any subset of R. $\theta(A) = (\underline{\theta}(\mathrm{A}), \overline{\theta}(\mathrm{A}))$ is called a rough set with repect to θ if $\underline{\theta}(\mathrm{A}) \neq \overline{\theta}(\mathrm{A})$.

Lemma 2.8. *For any approximation space* (R, θ) and $P \subseteq R$, *the following hold:*
(i) $\underline{\theta}(\mathrm{R}\backslash\mathrm{P}) = \mathrm{R}\backslash\overline{\theta}(\mathrm{P})$; (ii) $\overline{\theta}(\mathrm{R}\backslash\mathrm{P}) = \mathrm{R}\backslash\underline{\theta}(\mathrm{P})$; (iii) $\overline{\theta}(\mathrm{P}) = \left(\underline{\theta}(\mathrm{P}^c)\right)^c$; (iv) $\underline{\theta}(\mathrm{P}) = \left(\overline{\theta}(\mathrm{P}^c)\right)^c$
Proof. The proof is obvious and hence omitted.
The following theorem is immediate.

Theorem 2.9.[16] *Let* θ *and* ψ *be congruence relations on* R *and let* A *and* B *be nonempty subsets of* R. *Then*

(i) $\underline{\theta}(\mathrm{A}) \subseteq \mathrm{A} \subseteq \overline{\theta}(\mathrm{A})$
(ii) $\underline{\theta}(\emptyset) = \emptyset = \overline{\theta}(\emptyset)$
(iii) $\underline{\theta}(\mathrm{R}) = \mathrm{R} = \overline{\theta}(\mathrm{R})$
(iv) $\overline{\theta}(\mathrm{A} \cup \mathrm{B}) = \overline{\theta}(\mathrm{A}) \cup \overline{\theta}(\mathrm{B})$
(v) $\underline{\theta}(\mathrm{A} \cap \mathrm{B}) = \underline{\theta}(\mathrm{A}) \cap \underline{\theta}(\mathrm{B})$
(vi) $A \subseteq B$ implies $\underline{\theta}(\mathrm{A}) \subseteq \underline{\theta}(\mathrm{B})$ and $\overline{\theta}(\mathrm{A}) \subseteq \overline{\theta}(\mathrm{B})$
(vii) $\underline{\theta}(\mathrm{A} \cup \mathrm{B}) \supseteq \underline{\theta}(\mathrm{A}) \cap \underline{\theta}(\mathrm{B})$
(viii) $\overline{\theta}(\mathrm{A} \cap \mathrm{B}) \subseteq \overline{\theta}(\mathrm{A}) \cap \overline{\theta}(\mathrm{B})$
(ix) $\theta \subseteq \psi$ implies $\underline{\psi}(\mathrm{A}) \subseteq \underline{\theta}(\mathrm{A})$ and $\overline{\theta}(\mathrm{A}) \subseteq \overline{\psi}(\mathrm{A})$
(x) $(\overline{\theta \cap \psi})(\mathrm{A}) = \overline{\theta}(\mathrm{A}) \cap \overline{\psi}(\mathrm{A})$
(xi) $(\underline{\theta \cap \psi})(\mathrm{A}) \subseteq \underline{\theta}(\mathrm{A}) \cap \underline{\theta}(\psi)$
(xii) $\underline{\theta}\left(\underline{\theta}(\mathrm{A})\right) = \underline{\theta}(\mathrm{A})$
(xiii) $\overline{\theta}\left(\overline{\theta}(\mathrm{A})\right) = \overline{\theta}(\mathrm{A})$
(xiv) $\overline{\theta}\left(\underline{\theta}(\mathrm{A})\right) = \underline{\theta}(\mathrm{A})$
(xv) $\underline{\theta}\left(\overline{\theta}(\mathrm{A})\right) = \overline{\theta}(\mathrm{A})$.

Definition 2.10. Let A be any subset of R and (R, θ) be a rough approximation space. If $\underline{\theta}(\mathrm{A})$ and $\overline{\theta}(\mathrm{A})$ are ideals ,then $\underline{\theta}(\mathrm{A})$ is called a *lower and upper rough ideal* and $\overline{\theta}(\mathrm{A})$ is called an *upper rough ideal* of R, respectively. $\theta(A) = (\underline{\theta}(\mathrm{A}), \overline{\theta}(\mathrm{A}))$ is called *rough ideal* of R.

Theorem 2.11 *Let* θ *be a congruence relation on* R. *If* A *is a left (resp.right) ideal of* R, *then* $\overline{\theta}(\mathrm{A})$ *is a left (resp.right) ideal of* R.
Proof. Let $a, b \in \overline{\theta}(\mathrm{A})$. Then $[a]_\theta \cap A \neq \emptyset, [b]_\theta \cap A \neq \emptyset$. So there exist $x \in [a]_\theta \cap A$ and $y \in [b]_\theta \cap A$. Since $x, y \in A, x + y \in A$. Now $x + y \in [a]_\theta + [b]_\theta \subseteq [a + b]_\theta$. Therefore $[a + b]_\theta \cap A \neq \emptyset$ and this means that $a + b \in \overline{\theta}(\mathrm{A})$.

Again let $x \in \overline{\theta}(\mathrm{A})$ and $r \in R$. Then there exists $y \in [a]_\theta \cap A$ and $(y, x) \in \theta$. Since θ is congruence relation , $(xr, yr), (rx, ry) \in \theta$. This means that $xr, rx \in \overline{\theta}(\mathrm{A})$. Thus $\overline{\theta}(\mathrm{A})$ is an ideal of R.

Theorem 2.12. *Let θ be a congruence relation on R. If A is a left (resp.right) ideal of R and $\underline{\theta}(A)$ is nonempty, then $\underline{\theta}(A)$ is a left (resp.right) ideal of R.*

Proof. Let $a, b \in \underline{\theta}$(A). Then $[a]_\theta, [b]_\theta \subseteq A$. Consider $[a+b]_\theta \subseteq [a]_\theta + [b]_\theta \subseteq A + A \subseteq A$. Thus $a + b \in \underline{\theta}$(A).

Again let $a \in \underline{\theta}$(A) and $r \in R$. Consider, $[ar]_\theta \subseteq [a]_\theta . [r]_\theta \subseteq AR \subseteq A$ and $[ra]_\theta \subseteq [r]_\theta . [a]_\theta \subseteq RA \subseteq A$. Thus $\underline{\theta}$(A) is an ideal of R.

Corollary 2.13. *Let θ be a congruence relation on R. If A is an ideal of R and $\underline{\theta}(A)$ is nonempty, then $\theta(A) = (\underline{\theta}(A), \overline{\theta}(A)$ is a rough ideal of R.*

Lemma 2.14. *If I and J are ideals of R and $\underline{\theta}(I \cap J)$ is a nonempty set, then $\left(\underline{\theta}(I \cap J), \overline{\theta}(I \cap J)\right)$ is a rough ideal of R.*

Theorem 2.15. *Let φ be an epimorphism of a semiring R_1 to a semiring R_2 and let θ_2 be a congruence relation on R_2. Then*

(i) $\theta_1 = \{(a, b) \in R_1 \times R_2 | (\varphi(a), \varphi(b)) \in \theta_2\}$ *is a congruence relation.*

(ii) *If θ_2 is complete and φ is 1-1, then θ_1 is complete.*

(iii) $\varphi\left(\overline{\theta_1}(A)\right) = \overline{\theta_2}(\varphi(A))$

(iv) $\varphi(\underline{\theta_1}(A)) \subseteq \underline{\theta_2}(\varphi(A))$

(v) *If φ* is *1-1 , then* $\varphi\left(\underline{\theta_1}(A)\right) = \underline{\theta_2}(\varphi(A))$

Proof. (i) Let $(a, b) \in \theta_1$and $x \in R_1$. Then $(\varphi(a), \varphi(b)) \in \theta_2$. Since θ_2 is a congruence relation ,we have $\left(\varphi(a) + \varphi(x), \varphi(b) + \varphi(x)\right)$, $\left(\varphi(x) + \varphi(a), \varphi(x) + \varphi(b)\right)$, $\left(\varphi(a).\varphi(x), \varphi(b).\varphi(x)\right)$ and $\left(\varphi(x).\varphi(a), \varphi(x).\varphi(b)\right)$ are in θ_2. φ being homomorphism , $(\varphi(a+x), \varphi(b+x))$, $(\varphi(x+a), \varphi(x+b))$, $(\varphi(ax), \varphi(bx))$ and $(\varphi(xa), \varphi(xb))$ are in θ_2. Again since φ being onto, $(a+x, b+x)$, $(x+a, x+b)$, (ax, bx), $(xa, xb)$are in θ_1. Thus θ_1 is congruence relation in R_1.

(ii) Let θ_2 be complete . Assume that $z \in [ab]_{\theta_1}$. Then $(ab, z) \in \theta_1$. By definition of θ_2, $\left(\varphi(ab), \varphi(z)\right) \in \theta_2$. Hence

$$\begin{aligned} \varphi(z) &\in [\varphi(ab)]_{\theta_2} \\ &= [\varphi(a).\varphi(b)]_{\theta_2} \\ &= [\varphi(a)]_{\theta_2}.[\varphi(b)]_{\theta_2} \end{aligned}$$

Since $\varphi(z) \in [\varphi(a)]_{\theta_1}.[\varphi(b)]_{\theta_2}$, there exist $x, y \in R_1$ such that

$$\begin{aligned} \varphi(z) &= \varphi(x).\varphi(y) \\ &= \varphi(xy), \varphi(x) \in [\varphi(a)]_{\theta_2}, \varphi(y) \in [\varphi(b)]_{\theta_2} \end{aligned}$$

Since φ is 1-1 and by definition of θ_1, $z = xy$ and $x \in [a]_{\theta_1}$, $y \in [b]_{\theta_1}$. Thus $z \in [a]_{\theta_1}.[b]_{\theta_1}$ and therefore $[ab]_{\theta_1} \subseteq [a]_{\theta_1}.[b]_{\theta_1}$. By Lemma 2.6, $[a]_{\theta_1}.[b]_{\theta_1} \subseteq [ab]_{\theta_1}$. Hence $[ab]_{\theta_1} = [a]_{\theta_1}.[b]_{\theta_1}$.

Again suppose $z \in [a+b]_{\theta_1}$.In the similar manner, one can get $[a+b]_{\theta_1} = [a]_{\theta_1} + [b]_{\theta_1}$. Thus θ_1 is complete.

(iii) Let $y \in \varphi\left(\overline{\theta_1}(A)\right)$. Then there exists $x \in \overline{\theta_1}(A)$ such that $y = \varphi(x)$. This implies that $[x]_{\theta_1} \cap A \neq \emptyset$ and so there exists $a \in [x]_{\theta_1} \cap A$. Then $\varphi(a) \in \varphi(A)$ and $(a, x) \in \theta_1$ implies $(\varphi(a), \varphi(x)) \in \theta_2$. So $\varphi(a) \in [\varphi(x)]_{\theta_2}$. Thus $[\varphi(x)]_{\theta_2} \cap \varphi(A) \neq \emptyset$. This implies that $y = \varphi(x) \in \overline{\theta_2}(\varphi(A))$ and so

$$\varphi(\overline{\theta_1}(A)) \subseteq \overline{\theta_2}(\varphi(A)) \quad (1)$$

Again let $z \in \overline{\theta_2}(\varphi(A))$, there exists $x \in R_1$ such that $z = \varphi(x)$. Hence $[\varphi(x)]_{\theta_2} \cap \varphi(A) \neq \emptyset$. So there exists $a \in A$ such that $\varphi(a) \in \varphi(A)$ and $\varphi(a) \in [\varphi(x)]_{\theta_2}$. By definition of θ_1, we have $a \in [x]_{\theta_1}$. Thus $[x]_{\theta_1} \cap A \neq \emptyset$, which implies $x \in \overline{\theta_1}(A)$ and so $z = \varphi(x) \in \varphi(\overline{\theta_1}(A))$. It means that

$$\overline{\theta_2}(\varphi(A)) \subseteq \varphi(\overline{\theta_1}(A)) \quad (2)$$

From (1) and (2) the conclusion follows.

(iv) Let $y \in \varphi\left(\underline{\theta_1}(A)\right)$. Then there exists $x \in \underline{\theta_1}(A)$ such that $\varphi(x) = y$ and so we have $[x]_{\theta_1} \subseteq A$. Again let $b \in [y]_{\theta_2}$. Then there exists $a \in R_1$ such that $\varphi(a) = b$ and $\varphi(a) \in [\varphi(x)]_{\theta_2}$. Hence $a \in [x]_{\theta_1} \subseteq A$ and so $b = \varphi(a) \in \varphi(A)$. Thus $[y]_{\theta_2} \subseteq \varphi(A)$. This implies that $y \in \underline{\theta_2}(\varphi(A))$ and so we have $\varphi\left(\underline{\theta_1}(A)\right) \subseteq \underline{\theta_2}(\varphi(A))$.

(v) Let $y \in \underline{\theta_2}(\varphi(A))$. Then there exists $x \in R_1$ such that $\varphi(x) = y$ and $[\varphi(x)]_{\theta_2} \subseteq \varphi(A)$. Let $a \in [x]_{\theta_1}$. Then $\varphi(a) \in [\varphi(x)]_{\theta_2}$ and so $a \in A$. Thus $[x]_{\theta_1} \subseteq A$ and $x \in \underline{\theta_1}(A)$. Hence $y \in \varphi(x) \in \varphi\left(\underline{\theta_1}(A)\right)$ and so we have $\underline{\theta_2}(\varphi(A)) \subseteq \varphi\left(\underline{\theta_1}(A)\right)$. By (iv), we have

$\varphi\left(\underline{\theta_1}(A)\right) = \underline{\theta_2}(\varphi(A))$.

Theorem 2.16.[13] *Let θ be congruence relation on R and I be a subset of R. I is a k-ideal(resp. left k-ideal, right k- ideal) of R if and only if $\theta(I)$ is a rough k-ideal (resp. left k-ideal, right k- ideal) of R.*

Lemma 2.17.[13] *Let θ be any congruence relation on R and A and B be any subsets of R. If A and B are respectively, right and left k-ideals of R then*

(i) $\widehat{\overline{\theta}(A)\overline{\theta}(B)} \subseteq \overline{\theta}(A) \cap \overline{\theta}(B)$.

(ii) $\widehat{\underline{\theta}(A)\,\underline{\theta}(B)} \subseteq \underline{\theta}(A) \cap \underline{\theta}(B)$.

3 Rough Quasi-ideals in Regular Semi rings

In this section we study the concept of rough quasi-ideal and rough quasi-k-ideal in semiring R. We find the equivalent conditions under which a semiring R is regular.

Definition 3.1 A sub semiring Q is called a *quasi-ideal* of R if $QR \cap RQ \subseteq Q$.
A quasi-ideal Q is called a *quasi-k-ideal* of R if $\hat{Q} = Q$.

Lemma 3.2 *Let θ be any congruence relation on R. Then for any right k-ideal A and left k-ideal B of R, $\theta(A) \cap \theta(B)$ is a rough quasi k-ideal of R.*

Proof. Let A and B be any right k-ideal and left k-ideal of R respectively. Then by theorem 2.16, $\theta(A)$ is a rough right k-ideal and $\theta(B)$ is rough left k-ideal of R. Now

$$\begin{aligned}(\overline{\theta}(A) \cap \overline{\theta}(B))R \cap R(\overline{\theta}(A) \cap \overline{\theta}(B)) &\subseteq \overline{\theta}(A)R \cap R\overline{\theta}(B)\\ &\subseteq \overline{\theta}(AR) \cap \overline{\theta}(RB)\\ &\subseteq \overline{\theta}(A) \cap \overline{\theta}(B)\end{aligned}$$

and so $\overline{\theta}(A) \cap \overline{\theta}(B)$ is a quasi-k-ideal of R. Similarly we can show that $\underline{\theta}(A) \cap \underline{\theta}(B)$ is a quasi-k-ideal of R. Thus $\theta(A) \cap \theta(B)$ is rough quasi-k-ideal of R.

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Lemma 3.3. *Let θ be a complete congruence relation on R. If Q is a quasi-ideal of R and $\underline{\theta}(Q)$ is nonempty then $\theta(Q) = (\underline{\theta}(Q), \overline{\theta}(Q))$ is a rough quasi-ideal of R.*

Proof. Let $a, b \in \overline{\theta}(Q)$. Then there exists $x \in [a]_\theta \cap Q$ and $y \in [b]_\theta \cap Q$. Since $x, y \in Q, x + y \in Q$ and by Lemma 2.6(i) $x + y \in [a]_\theta + [b]_\theta \subseteq [a + b]_\theta$. Since $x + y \in Q$ and $x + y \in [a + b]_\theta$, $a + b \in \overline{\theta}(Q)$. Again suppose that $a, b \in \underline{\theta}(Q)$. Then $[a]_\theta \subseteq Q$ and $[b]_\theta \subseteq Q$. This implies that $a + b \in [a]_\theta + [b]_\theta \subseteq Q + Q \subseteq Q$.

Hence $a + b \in \underline{\theta}(Q)$. Let $x \in \overline{\theta}(Q)R$. Then $x = ar_1$ with $a \in \overline{\theta}(Q)$, $r_1 \in R$. Now $[a]_\theta \cap Q \neq \emptyset$. Thus there exists $y \in [a]_\theta \cap Q$ such that $y \in [a]_\theta$ and $y \in Q$. Then $(a, y) \in \theta$ and $y \in Q$. Since Q is a quasi-ideal of R, $yr_1 \in Q$. Now $(a, y) \in \theta$,implies , by congruence of θ, $(ar_1, yr_1) \in \theta$ which means $yr_1 \in [ar_1]_\theta$ and $yr_1 \in Q$, hence we have $[a]_\theta \cap Q \neq \emptyset$ and therefore $x = ar_1 \in \overline{\theta}(Q)$. This shows that $\overline{\theta}(Q)R \subseteq \overline{\theta}(Q)$.

Consider $\overline{\theta}(Q)R \cap R\overline{\theta}(Q) \subseteq \overline{\theta}(Q)R \subseteq \overline{\theta}(Q)$.

This means that $\overline{\theta}(Q)$is a quasi-ideal of R.

Since Q is a quasi-ideal of R and by Theorem2.9(iii),

$$\begin{aligned}\big(\underline{\theta}(Q)R\big) \cap (R\underline{\theta}(Q)) &\subseteq \underline{\theta}(QR) \cap \underline{\theta}(RQ)\\ &\subseteq \underline{\theta}(QR \cap RQ)\\ &\subseteq \underline{\theta}(Q).\end{aligned}$$

Thus $\underline{\theta}(Q)$ is a quasi-ideal of R. Combining $\overline{\theta}(Q)$ and $\underline{\theta}(Q)$ are quasi-ideals, $\theta(Q)$ is a quasi-ideal of R.

Theorem 3.4 *Let θ be a congruence relation on R and let A be a left(right) ideal of R and $\underline{\theta}(A)$ be non-empty then $\theta(A)$ is a rough quasi-ideal of R.*

Proof. Let A be a left ideal of R. Then by Theorem 2.9(iii) $\overline{\theta}(A)R = \overline{\theta}(A)\overline{\theta}(R) \subseteq \overline{\theta}(AR) \subseteq \overline{\theta}(A)$. So, $\overline{\theta}(A)R \cap R\overline{\theta}(A) \subseteq \overline{\theta}(A)R \subseteq \overline{\theta}(A)$. Thus $\overline{\theta}(A)$ is a quasi-ideal of R.

Again, $\underline{\theta}(A)R = \underline{\theta}(A)\underline{\theta}(R) \subseteq \underline{\theta}(AR) \subseteq \underline{\theta}(A)$ and $\underline{\theta}(A)R \cap R\underline{\theta}(A) \subseteq \underline{\theta}(A)R \subseteq \underline{\theta}(A)$. Thus $\underline{\theta}(A)$ is a quasi-ideal of R. Therefore $\theta(A)$ is a rough quasi-ideal of R.

Theorem 3.5. *Let θ be a congruence relation on R. If a sub semiring Q is a quasi-k-ideal, then $\theta(Q)$ is a rough quasi-k-ideal of R.*

Proof. Let Q be a quasi-k-ideal of R. Then by Lemma 3.3, $\theta(Q)$ is a rough quasi-ideal of R. Let $a, a + b \in \overline{\theta}(Q)$. Then $[a]_\theta \cap Q \neq \emptyset$ and $[a + b]_\theta \cap Q \neq \emptyset$. Then there exists $x, y \in R$ such that $\in [a]_\theta$, $x \in Q$ and $y \in [a + b]_\theta$ implies $a + b \in [y]_\theta \subseteq Q, a + b \in Q$, Q being k-ideal, $b \in Q$. Thus $[b]_\theta \cap Q \neq \emptyset$ and this shows that $b \in \overline{\theta}(Q)$ and thus $\overline{\theta}(Q)$ is a quasi-*k*-ideal of R.

Again let $a, a + b \in \underline{\theta}(Q)$. Then $[a]_\theta \subseteq Q$ and $[a + b]_\theta \subseteq Q$. This implies that $a, a + b \in Q$. Since Q is a quasi-*k*-ideal of R, $b \in Q$ and $[b]_\theta \subseteq Q$. Thus $\underline{\theta}(Q)$ is a *k*-ideal of R. Thus $\theta(Q)$ is a rough quasi-*k*-ideal of R.

A band is a semigroup in which every element is an idempotent. A commutative band is called a semilattice.

Definition 3.6. A semiring R is called a *regular semiring* if for every $a \in R$ there exist $x, y \in R$ such that $a + axb = aya$. For a semilattice $(R, +)$,

$$\begin{aligned}a + axb = aya &\Rightarrow a + axa + (axa + aya) = aya + (axa + aya)\\ &\Rightarrow a + axa + aya = axa + aya\\ &\Rightarrow a + a(x + y)a = a(x + y)a\end{aligned}$$

This shows that R is regular iff for all $a \in R$ there exists $x \in R$ such that $a + axa = axa$.

Theorem 3.7. *Let θ be a convergence relation on R. R is regular if and only if $\theta(\widehat{A)\theta(}B) = \theta(A) \cap \theta(B)$ for any right k-ideal A and left k-ideal B of R.*
Proof. By Theorem 2.16 , $\theta(A)$ and $\theta(B)$ are rough k-ideal and rough left k-ideal of R respectively and by Lemma 2.17(i), $\overline{\theta}(\widehat{A)\overline{\theta}(}B) \subseteq \overline{\theta}(A) \cap \overline{\theta}(B)$. Now $a \in \overline{\theta}(A) \cap \overline{\theta}(B)$, hence there exists $x \in R$ such that $a + axa = axa$. Now $(ax)a \in \overline{\theta}(A)\overline{\theta}(B)$ implies $a \in \overline{\theta}(\widehat{A)\overline{\theta}(}B)$ and so $\overline{\theta}(A)$and $\overline{\theta}(B) \subseteq \overline{\theta}(\widehat{A)\overline{\theta}(}B)$. Thus $\overline{\theta}(\widehat{A)\overline{\theta}(}B) = \overline{\theta}(A) \cap \overline{\theta}(B)$.Similarly $\underline{\theta}(\widehat{A)\underline{\theta}(}B) = \underline{\theta}(A) \cap \underline{\theta}(B)$. Thus $\theta(\widehat{A)\theta(}B) = \theta(A) \cap \theta(B)$.
Conversely, assume that $\theta(\widehat{A)\theta(}B) = \theta(A) \cap \theta(B)$ for any right k-ideal A and left k-ideal B of R. This implies $\overline{\theta}(\widehat{A)\overline{\theta}(}B) = \overline{\theta}(A) \cap \overline{\theta}(B)$ and $\underline{\theta}(\widehat{A)\underline{\theta}(}B) = \underline{\theta}(A) \cap \underline{\theta}(B)$. Let $a \in R$.

$$r(a)_k = \{u \in R | u + a + ar = a + ar\} \subseteq \overline{\theta}(r(a)_k)$$
$$l(a)_k = \{v \in R | v + a + ra = a + ar\} \subseteq \overline{\theta}(l(a)_k)$$
$$a \in \overline{\theta}(r(a)_k) \cap \overline{\theta}(l(a)_k) = \overline{\theta}(r(a)\widehat{_k)\overline{\theta}(}l(a)_k).$$

Then there exists $x, y \in R$ and $q \in \overline{\theta}(r(a)_k) \cap \overline{\theta}(l(a)_k)$ such that $q + xa + a = xa + a$ and $q + ay + a = ay + a$ by $L_k(a) \subseteq \overline{\theta}(L_k(a))$ and $R_k(a) \subseteq \overline{\theta}(R_k(a))$. Thus

$$\begin{aligned} a + qrq = qrq &\Rightarrow a + (q + ay + a)r(q + xa + a) = (q + ay + a)r(q + xa + a) \\ &\Rightarrow a + (ay + a)r(xa + a) = (ay + a)r(xa + a) \\ &\Rightarrow a + a(yrx + yr + rx + r)a = a(yrx + yr + rx + r)a \\ &\Rightarrow a + ata = ata, t = yrx + yr + rx + r \in R. \end{aligned}$$

This shows that R is a regular semiring.

Theorem 3.8. *Let θ be a congruence relation on R. The following conditions are equivalent:*

(i) *R is regular*

(ii) *$\theta(Q) = \theta(\widehat{Q)R\theta}(Q)$ for every quasi-k-ideal Q of R.*

Proof. (i)⇒(ii): Let Q be a quasi-k-ideal of R. By Theorem 3.5, $\theta(Q)$ is a rough quasi-k-ideal of R. Now $\theta(Q)R\theta(Q) \subseteq \theta(Q)RR \subseteq \theta(Q)R$ and $\theta(Q)R\theta(Q) \subseteq RR\theta(Q) \subseteq R\theta(Q)$. Since $\theta(Q)$ is a rough quasi-k-ideal of R, $\theta(Q)R\theta(Q) \subseteq \theta(Q)R \cap R\theta(Q)$. $\theta(\widehat{Q)R\theta}(Q) \subseteq \hat{\theta}(Q) = \theta(Q)$, because $\theta(Q)$ is a rough quasi-k-ideal of R. Let $a \in \theta(Q)$. Since R is regular , there exists $x \in R$ such that $a + axa = axa$. Then $axa \in \theta(Q)R\theta(Q)$ implies that $a \in \theta(\widehat{Q)}R\theta(Q)$. This implies $\theta(Q) \subseteq \theta(\widehat{Q)R\theta}(Q)$. Thus $\theta(Q) = \theta(\widehat{Q)R\theta}(Q)$.
(ii)⇒(i):
Let $a \in R$. Then $\theta(Q) = \theta\big(L_k(a)\big) \cap \theta\big(R_k(a)\big)$ is a rough quasi-k-ideal of R, by Lemma 3.2. Then $a \in \overline{\theta}(L_k(a)) = \overline{\theta}(L_k\widehat{(a))R\overline{\theta}}(R_k(a))$. This means that there exists $q_1, q_2, q_3, q_4 \in \overline{\theta}(L_k(a))$ and $r_1, r_2 \in R$, so that $a + q_1 r q_2 = q_3 r_2 q_4$.
This implies that
$a + (q_1 + q_2 + q_3 + q_4)(r_1 + r_2)(q_1 + q_2 + q_3 + q_4) = (q_1 + q_2 + q_3 + q_4)(r_1 + r_2)(q_1 + q_2 +$ *q3+q4*. This means that *a+qrq=qrq,* where
$q = (q_1 + q_2 + q_3 + q_4) \in \overline{\theta}(Q) = \overline{\theta}(L_k(a)) \cap \overline{\theta}(R_k(a))$ and $r = r_1 + r_2 \in R$.
Then there exist $u \in \overline{\theta}(r(a)_k)$ and $v \in \overline{\theta}(l(a)_k)$ so that $a + uv = uv$. This implies that $a + (a + ar)(ra + a) = (a + ar)(ra + a)$. Hence $a + aza = aza$ for some $z \in R$. This means that R is regular.

4 Conclusion

The purpose of this paper is to build a connection between rough set theory and regular semiring. We have introduced the notions of lower and upper approximation subsets of a semiring and characterized the quasi and quasi-k-ideals of a regular semiring through lower and upper approximations. This definition and results can be extended to other algebraic structures such as rings and modules.

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